EXERCISE – IV ADVANCED SUBJECTIVE QUESTIONS

1. $\int_0^{2\pi} e^x \cos\left(\frac{\pi}{4}+\frac{x}{2}\right)dx$

2. $\int_0^{\pi/4} \frac{\cos x-\sin x}{10+\sin 2x}dx$

3. $\int_0^{\pi} \frac{(ax+b)\sec x\tan x}{4+\tan^2 x}dx$ (a, b > 0)

4. $\int_0^{\pi} \frac{(2x+3)\sin x}{(1+\cos^2 x)}dx$

5. Show that $\int_0^{p+q\pi} |\cos x|\,dx = 2q + \sin p$ where $q \in N$ & $-\frac{\pi}{2} < p < \frac{\pi}{2}$

6. $\int_0^{1} \frac{\sin^{-1}\sqrt{x}}{x^2-x+1}dx$

7. $\int_0^{\pi/2} \tan^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]dx$

8. $\int_{\sqrt{\frac{3a^2+b^2}{2}}}^{\sqrt{\frac{a^2+b^2}{2}}} \frac{x.dx}{(x^2-a^2)(b^2-x^2)}$

9. Comment upon the nature of roots of the quadratic equation $x^2 + 2x = k + \int_0^1 |t+k|\,dt$ depending on the value of $k \in R$.

10. $\int_0^{2a} x\sin^{-1}\left[\frac{1}{2}\sqrt{\frac{2a-x}{a}}\right]dx$

11. Let $u=\int_0^{\pi/4}\left(\frac{\cos x}{\sin x+\cos x}\right)^2 dx$ and $v=\int_0^{\pi/4}\left(\frac{\sin x+\cos x}{\cos x}\right)^2 dx$. Find the value of $\frac{v}{u}$.

12. $\int_0^{2\pi} \frac{x^2\sin x}{8+\sin^2 x}dx$

13. $\int_0^{\pi/4} \frac{x^2(\sin 2x-\cos 2x)}{(1+\sin 2x)\cos^2 x}dx$

14. Prove that $\int_0^{x}\left(\int_0^{u} f(t)\,dt\right)du = \int_0^{x} f(u).(x-u)\,du$

15. $\int_0^{\pi} \frac{dx}{(5+4\cos x)^2}$

16. Evaluate $\int_0^{1} \ell n(\sqrt{1-x}+\sqrt{1+x})\,dx$

17. $\int_1^{16} \tan^{-1}\sqrt{\sqrt{x}-1}\,dx$

18. $\lim\limits_{n\to\infty} n^2 \int_{-1/n}^{1/n} (2006\sin x+2007\cos x)\,|x|\,dx$.

19. Show that $\int_0^{\infty} f\left(\frac{a}{x}+\frac{x}{a}\right).\frac{\ln x}{x}\,dx = \ln a . \int_0^{\infty} f\left(\frac{a}{x}+\frac{x}{a}\right).\frac{dx}{x}$

20. Evaluate the definite integral, $\int_{-1}^{1} \frac{(2x^{332}+x^{998}+4x^{1668}.\sin x^{691})}{1+x^{666}}dx$.

21. For $a \geq 2$, if the value of the definite integral $\int_0^{\infty} \frac{dx}{a^2+\left(x-\frac{1}{x}\right)^2}$ equals $\frac{\pi}{5050}$. Find the value of a.

22. Evaluate : $\int_0^1 e^{\ell n \tan^{-1} x}.\sin^{-1}(\cos x)\,dx$.

23. If the derivative of f(x) w.r.to x is $\frac{\cos x}{f(x)}$ then show that f(x) is a periodic function.

24. Find the range of the function,

$f(x) = \int_{-1}^{1} \frac{\sin x\, dt}{1-2t\cos x+t^2}$.

25. A function r is defined in [–1, 1] as

$f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$; $x \neq 0$; f(0) = 0 ; f(1/π) = 0.
Discuss the continuity and derivability of f at x = 0.

26. Let $f(x) = \begin{cases} -1 & \text{if } -2 \le x \le 0 \\ |x-1| & \text{if } 0 < x \le 2 \end{cases}$ and $g(x) = \int_{-2}^{x} f(t)\,dt$.
Test the continuity and differentiability of g(x) in (–2, 2).

27. Prove the inequalities

(a) $\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \frac{\pi\sqrt{2}}{8}$

(b) $2e^{-1/4} < \int_0^2 e^{x^2-x}\,dx < 2e^2$

(c) $a < \int_0^{2\pi} \frac{dx}{10+3\cos x} < b$ then find a & b.

(d) $\frac{1}{2} \le \int_0^2 \frac{dx}{2+x^2} \le \frac{5}{6}$

28. If $y = \frac{1}{a}\int_0^x f(t).\sin a(x-t)\,dt$ then prove that $\frac{d^2y}{dx^2} + a^2 y = f(x)$.

29. If $y = x^{\int_1^x \ell n\, t\, dt}$, find $\frac{dy}{dx}$ at x = e.

30. If $f(x) = x + \int_0^1 [xy^2 + x^2 y]\, f(y)\,dy$ where x and y are independent variable, Find f(x).

31. (a) Let $f(x) = x^c \,.\, e^{2x}$ & let $f(x) = \int_0^x e^{2t}.(3t^2+1)^{1/2}\,dt$.
For a certain value of 'c', the limit of $\frac{f'(x)}{g'(x)}$ as $x \to \infty$ is finite and non zero. Determine the value of 'c' and the limit.

(b) Find the constants 'a' (a > 0) and 'b' such that,

$$\lim_{x\to 0} \frac{\int_0^x \frac{t^2\,dt}{\sqrt{a+t}}}{bx-\sin x} = 1.$$

32. Evaluate : $\lim_{x\to+\infty} \frac{d}{dx}\int_{2\sin\frac{1}{x}}^{3\sqrt{x}} \frac{3t^4+1}{(t-3)(t^2+3)}\,dt$

33. Supose g(x) is the inverse of f(x) and f(x) has a domain $x \in [a, b]$. Given f(a) = α and f(b) = β, then find the value of $\int_a^b f(x)\,dx + \int_\alpha^\beta g(y)\,dy$ in terms of a, b, α and β.

34. Evaluate

(a) $\lim_{n\to\infty} \left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right).....\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$

(b) $\lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{n+1}+\frac{2}{n+2}+......+\frac{3n}{4n}\right]$

(c) $\lim_{n\to\infty} \left[\frac{n!}{n^n}\right]^{1/n}$

(d) For positive integers n, let

$A_n = \frac{1}{n}\{(n + 1) + (n + 2) + ... (n + n)\}$,
$B_n = \{(n + 1)(n + 2) .... (n + n)\}^{1/n}$.

If $\frac{A_n}{B_n} = \frac{ae}{b}$ where a, b $\in$ N and relatively prime find the value of (a + b).

35. Prove that sin x+sin 3x+ sin 5x +...+ sin (2k–1) x

$= \frac{\sin^2 kx}{\sin x}$, $k \in N$ and hence prove that,

$$\int_0^{\pi/2} \frac{\sin^2 kx}{\sin x}\, dx = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots\ldots + \frac{1}{2k-1}.$$

36. Solve the equation for y as a function of x, satisfying

$x \cdot \int_0^x y(t)\, dt = (x+1) \int_0^x t \cdot y(t)\, dt$, where $x > 0$, given $y(1) = 1$.

37. Prove that

(a) $I_{m,n} = \int_0^1 x^m \cdot (1-x)^n\, dx = \frac{m!\,n!}{(m+n+1)!}$ $m, n \in N$.

(b) $I_{m,n} = \int_0^1 x^m \cdot (\ell n\, x)^n dx = (-1)^n \frac{n!}{(m+1)^{n+1}}$ $m, n \in N$

38. Find a positive real valued continuously differentiable functions f on the real line such that for all x

$f^2(x) = \int_0^x (f(t))^2 + (f'(t))^2\, dt + e^2$

39. Let f(x) be a continuously differentiable function

then prove that, $\int_1^x [t]\, f'(t)\, dt = [x].f(x) - \sum_{k=1}^{[x]} f(k)$

(where [$*$] denotes the greatest integer function and $x > 1$)

40. Let $f(x) = \int_{-1}^x \sqrt{4+t^2}\, dt$ and $G(x) = \int_x^1 \sqrt{4+t^2}\, dt$

then compute the value of (FG)' (0) where dash denotes the derivative.

41. Show that for a continuously thrice differentiable function f(x)

$$f(x) - f(0) = xf'(0) + \frac{f''(0).x^2}{2} + \frac{1}{2}\int_0^x f'''(t)(x-t)^2\, dt$$

42. Let f and g be function that are differentiable for all real numbers x and that have the following properties

(i) $f'(x) = f(x) - g(x)$ **(ii)** $g'(x) = g(x) - f(x)$

(iii) $f(0) = 5$ **(iv)** $g(0) = 1$

(a) Prove that $f(x) + g(x) = 6$ for all x.

(b) Find f(x) and g(x).